

# CHARACTERISTIC POLYNOMIALS OF CENTRAL SIMPLE ALGEBRAS

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**ABSTRACT.** We characterize characteristic polynomials of elements in a central simple algebra. We also give an account for the theory of rational canonical forms for separable linear transformations over a central division algebra, and a description of separable conjugacy classes of the multiplicative group.

## 1. INTRODUCTION

Consider a  $p$ -adic local field  $F$  (i.e. a finite field extension of  $\mathbb{Q}_p$ ) and the connected reductive groups  $G$  and  $G'$  over  $F$  of multiplicative groups of the matrix algebra  $\text{Mat}_n(F)$  and a central division algebra  $D$  over  $F$  of degree  $n$ . The local Jacquet-Langlands correspondence states that there is a one-to-one correspondence

$$E^2(G') \simeq E^2(G)$$

between the sets  $E^2(G')$  and  $E^2(G)$  of essentially square-integral irreducible smooth representations of  $G'$  and  $G$  [1, p. 34]. The correspondence is characterized by the following character identity

$$(1.1) \quad \chi_{\pi'}(g') = (-1)^{n-1} \chi_{\pi}(g), \quad \forall \{g\} \text{ corresponds to } \{g'\}$$

if  $\pi$  corresponds to  $\pi'$ , where  $\chi_{\pi'}$  (resp.  $\chi_{\pi}$ ) is the character of  $\pi'$  (resp.  $\pi$ ), which is a locally constant central function defined at least on the open set  $G'(F)^{\text{reg}}$  (resp.  $G(F)^{\text{reg}}$ ) of regular semi-simple elements. The conjugacy class  $\{g\}$  corresponding to the conjugacy class  $\{g'\}$  means that they share the same characteristic polynomial. Clearly not every regular semi-simple conjugacy class in  $G(F)$  corresponds to a class in  $G'(F)$ . In this Note we consider the following basic question in full generality:

**(Q)** Let  $A$  be a finite-dimensional central simple algebra over an arbitrary field  $F$ . Which polynomials of degree  $\deg(A)$  are the characteristic polynomials of elements in  $A$ ?

We recall some basic definitions for central simple algebras; see [5].

**Definition 1.** Let  $A$  be a (f.d.) central simple algebra over a field  $F$ .

(1) The *degree*, *capacity*, and *index* of  $A$  are defined as

$$\deg(A) := \sqrt{[A : F]}, \quad c(A) := n, \quad i(A) := \sqrt{[\Delta : F]},$$

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respectively if  $A \cong \text{Mat}_n(\Delta)$ , where  $\Delta$  is a division algebra over  $F$ , which is uniquely determined by  $A$  up to isomorphism. The algebra  $\Delta$  is also called the *division part* of  $A$ .

- (2) For any element  $x \in A = \text{Mat}_n(\Delta)$ , the *characteristic polynomial* of  $x$  is defined to be the characteristic polynomial of the image of  $x$  in  $\text{Mat}_{nd}(\bar{F})$  under a map

$$A \rightarrow A \otimes_F \bar{F} \xrightarrow{\rho} \text{Mat}_{nd}(\bar{F}),$$

where  $\bar{F}$  is an algebraic closure of  $F$  and  $d$  is the degree of  $\Delta$ . This polynomial is independent of the choice of the isomorphism  $\rho$  and it is defined over  $F$ . Denote by  $f_x(t)$  the characteristic polynomial of  $x$ .

The question **(Q)** would become more interesting when one specializes  $F$  as a global field. When  $A$  is a quaternion algebra over a number field  $F$ , the answer is well-known. It is easier to treat the case when the polynomial  $f(t) := t^2 + at + b$  factorizes in  $F[t]$  (the answer is yes if and only if  $f(t)$  has double roots or  $A$  splits). So we may consider the case when  $f(t)$  is irreducible. Let  $K$  be a splitting field of  $f(t)$ , which is unique up to isomorphism. Then  $f(t)$  is a characteristic polynomial of  $A$  if and only if there is a  $F$ -algebra embedding of  $K$  in  $A$ . Then one can check the latter condition easily by the local-global principle (cf. Prasad-Rapinchuk [4, Proposition A.1]) which asserts that this is equivalent to that whenever a place  $v$  of  $F$  ramified in  $A$ , the  $F_v$ -algebra  $K_v := K \otimes_F F_v$  is a field, where  $F_v$  denotes the completion of  $F$  at the place  $v$ .

It is obvious that the question **(Q)** should land in the content of linear algebra when one realizes  $A = \text{Mat}_n(\Delta)$  as the algebra of  $\Delta$ -linear transformations on the  $\Delta$ -vector space  $\Delta^n$ . There are several books (for example the famous book by Jacobson [2]) on linear algebra that deal with vector spaces over division rings. However, we could not find one that deals with some core topics such as eigenvalues, eigenspaces, Jordan canonical forms and rational canonical forms over central division algebras. As these are seemingly missing in the literature, we give an account for the theory of rational canonical forms over a (f.d.) central division  $F$ -algebra  $\Delta$ . However, our theory is explicit only for *separable* linear transformations. We call a  $\Delta$ -linear transformation  $x$  on a (f.d.)  $\Delta$ -vector space  $V$  *separable* if its characteristic polynomial  $f_x(t)$  is the product of irreducible separable polynomials in  $F[t]$ . The restriction of the separability assumption is due to the separability assumption in a theorem of Cohen (cf. [3, Theorem 60, p. 205]) we apply. The main result of this Note (Theorem 4) determines which polynomial is a characteristic polynomial. After that we give an explicit description of separable conjugacy classes of the multiplicative group  $A^\times$  (Theorem 9).

From now on, we fix a finite-dimensional central simple algebra  $A$  over an arbitrary ground field  $F$ . Let  $A = \text{End}_\Delta(V)$ , where  $\Delta$  is the division part of  $A$  and  $V$  is a right vector space over  $\Delta$  of dimension  $n$ . Put  $d := \deg(\Delta)$ .

## 2. MINIMAL POLYNOMIALS

Let  $x$  be an element of  $A$ . The *minimal polynomial* of  $x$  is the unique monic polynomial  $m_x(t) \in F[t]$  of least degree such that  $m_x(x) = 0$  in  $A$ . Let  $F[x] \subset A$  be the  $F$ -subalgebra generated by the element  $x$ . One has  $F[x] = F[t]/(m_x(t))$

and  $V$  is an  $(F[x], \Delta)$ -bimodule. Since  $A = \text{End}_\Delta(V)$ , the space  $V$  is a faithful  $F[x]$ -module. When the ground division algebra  $\Delta$  is equal to  $F$ , the theory of rational canonical forms is nothing but the classification theorem for faithful  $F[x]$ -modules of  $F$ -dimension  $n$ . Therefore, the classification theorem for  $F[x]$ -faithful  $(F[x], \Delta)$ -bimodules of  $\Delta$ -dimension  $n$  is exactly the theory of rational canonical forms over  $\Delta$ .

**Lemma 2.**

- (1) *Let  $p(t) \in F[t]$  be an irreducible polynomial of positive degree. Then  $p(t)|m_x(t)$  if and only if  $p(t)|f_x(t)$ .*
- (2) *If  $m_x(t)$  is an irreducible polynomial  $p(t)$ , then  $f_x(t) = p(t)^a$ , where  $a := \deg(A)/\deg p(t)$ .*

PROOF. (1) This is a basic result in linear algebra when  $A$  is a matrix algebra over  $F$ . Suppose  $p(t)|f_x(t)$ . Let  $\alpha \in \bar{F}$  be a root of  $p(t)$ . Since  $A \otimes \bar{F}$  is a matrix algebra over  $\bar{F}$ , the basic result shows that  $(t - \alpha)|f_x(t)$  in  $\bar{F}[t]$  if and only if  $(t - \alpha)|m_x(t)$  in  $\bar{F}[t]$ . Therefore,  $(t - \alpha)|m_x(t)$  in  $\bar{F}[t]$  and hence  $p(t)|m_x(t)$  in  $F[x]$ .

(2) This follows from (1). ■

We write the minimal polynomial  $m_x(t) = \prod_{i=1}^s p_i(t)^{e_i}$  into the product of irreducible polynomials, where  $p_i(t) \in F[t]$  is a monic irreducible polynomial,  $p_i \neq p_j$  if  $i \neq j$ , and  $e_i$  is a positive integer. Put  $F_i := F[t]/(p_i(t))$  and  $\tilde{F}_i := F[t]/(p_i(t)^{e_i})$ . By the Chinese Remainder Theorem, one has

$$(2.1) \quad F[x] \simeq \prod_{i=1}^s \tilde{F}_i.$$

Note that  $F[x]$  is an Artinian ring. The decomposition (2.1) is nothing but the decomposition of  $F[x]$  into the product of local Artinian rings. Since  $\tilde{F}_i$  is a local Artinian ring with residue field  $F_i$ , it is a complete local Noetherian  $F$ -algebra with residue field  $F_i$ . By Cohen's theorem [3, Theorem 60, p. 205], there is a ring monomorphism  $s_i : F_i \rightarrow \tilde{F}_i$  such that  $\pi \circ s_i = \text{id}_{F_i}$  for all  $i$ , where  $\pi : \tilde{F}_i \rightarrow F_i$  is the natural projection. The cotangent space  $\mathfrak{m}_{\tilde{F}_i}/\mathfrak{m}_{\tilde{F}_i}^2$  of  $\tilde{F}_i$  is of  $F$ -dimension  $\deg p_i(t)$ , and hence it is a one-dimensional  $F_i$ -vector space. Let  $\varepsilon_i \in \tilde{F}_i$  be a generator of the maximal ideal  $\mathfrak{m}_{\tilde{F}_i}$  of  $\tilde{F}_i$ , one yields  $\tilde{F}_i = s_i(F_i)[\varepsilon_i]/(\varepsilon_i^{e_i})$ .

### 3. RATIONAL CANONICAL FORMS

We now come to the  $(F[x], \Delta)$ -bimodule structure on  $V$ , or equivalently, the (right)  $\Delta \otimes_F F[x]$ -module structure on  $V$ . The decomposition (2.1) gives a decomposition of  $V$  into  $\Delta$ -submodules  $V_i$  on which the  $F$ -algebra  $\tilde{F}_i$  acts faithfully:

$$(3.1) \quad V = \bigoplus_{i=1}^s V_i, \quad \dim_\Delta V_i =: n_i > 0, \quad \sum_{i=1}^s n_i = n.$$

We say an element  $y$  of  $A$  *separable* if its characteristic polynomial is the product of irreducible *separable* polynomials in  $F[t]$ . As  $A = \text{End}_\Delta(V)$ , a  $\Delta$ -linear transformation on  $V$  is called *separable* if it is separable as an element in  $A$ . Assume that  $x$  is separable. Then one can choose  $s_i$  so that  $F \subset s_i(F_i)$  by Cohen's theorem. Suppose that  $\Delta \otimes_F s_i(F_i) = \text{Mat}_{c_i}(\Delta_i)$ , where  $c_i$  (resp.  $\Delta_i$ ) is the capacity (resp.

the division part) of the central simple algebra  $\Delta \otimes_F s_i(F_i)$  over  $s_i(F_i)$ . Each  $V_i$  is a right  $\Delta \otimes_F \tilde{F}_i$ -module. One has

$$\Delta \otimes_F \tilde{F}_i = [\Delta \otimes_F s_i(F_i)] \otimes_{s_i(F_i)} \tilde{F}_i = \text{Mat}_{c_i}(\tilde{\Delta}_i), \quad \tilde{\Delta}_i := \Delta_i \otimes_{s_i(F_i)} \tilde{F}_i = \Delta_i[\varepsilon_i]/(\varepsilon_i^{e_i}).$$

By the Morita equivalence, we have  $V_i = W_i^{\oplus c_i}$ , where  $W_i$  is a  $\tilde{\Delta}_i$ -module. Since  $\Delta_i[\varepsilon_i]/(\varepsilon_i^{e_i})$  is a non-commutative PID, there is an isomorphism of  $\tilde{\Delta}_i$ -modules

$$(3.2) \quad W_i \simeq \bigoplus_{j=1}^{t_i} \Delta_i[\varepsilon_i]/(\varepsilon_i^{m_j})$$

for some positive integers  $1 \leq m_1 \leq \dots \leq m_{t_i} \leq e_i$ . As  $\tilde{F}_i$  acts faithfully on  $W_i$ , one has  $m_{t_i} = e_i$ . The decomposition (3.1) is exactly the decomposition of  $V$  into generalized eigenspaces in the classical case. Moreover, the decomposition (5) is exactly a decomposition of each generalized eigenspace into indecomposable invariant components. These give all information to make the rational canonical form for a suitable choice of basis as one does in linear algebra when  $\Delta = F$  (so far our theory is explicit only for separable  $\Delta$ -linear transformations). We leave the details to the reader.

*Remark 3.* One needs explicit information of  $\Delta, F$  and  $F_i$  in order to compute the capacity  $c_i$  and the division part  $\Delta_i$  of  $\Delta \otimes_F s_i(F_i)$  (or of  $\Delta \otimes_F F_i$ ). When  $F$  is a global field, these can be computed explicitly using Brauer groups over local fields [6] and the period-index relation [5, Theorem 32.19, p. 280]; see [7, Section 3] for details.

#### 4. WHICH POLYNOMIAL IS CHARACTERISTIC?

Let  $f(t) \in F[t]$  be a monic polynomial of degree  $nd$ . We will determine when  $f(t) = f_x(t)$  for some element  $x \in A$ . We say that  $f(t)$  is a *characteristic polynomial* of  $A$  if  $f(t) = f_x(t)$  for some element  $x \in A$ . Our main result of this Note is the following:

**Theorem 4.** *Let  $f(t) \in F[t]$  be a monic polynomial of degree  $nd$  and let  $f(t) = \prod_{i=1}^s p_i(t)^{a_i}$  be the factorization into irreducible polynomials. Put  $F_i := F[t]/(p_i(t))$ . Then  $f(t)$  is a characteristic polynomial of  $A$  if and only if for all  $i = 1, \dots, s$ , one has*

- (a)  $a_i \deg p_i(t) = n_i d$  for some positive integer  $n_i$ , and
- (b)  $[F_i : F] \mid n_i \cdot c(\Delta \otimes_F F_i)$ .

Suppose  $f(t) = f_x(t)$  for some  $x \in A$ . Then the minimal polynomial  $m_x(t)$  of  $x$  is equal to  $\prod_{i=1}^s p_i(t)^{e_i}$  for some positive integers  $e_i$  with  $e_i \leq a_i$ . Discussion in Section 3 shows that there is a decomposition of  $V$  into  $\Delta$ -subspaces  $V_i$  say of  $\Delta$ -dimension  $n_i$  on which the  $F$ -algebra  $\tilde{F}_i$  acts faithfully. Regarding  $x : V \rightarrow V$  as a  $\Delta$ -linear transformation, let  $x_i$  be the restriction of  $x$  on the invariant subspace  $V_i$ . Then we have

$$(4.1) \quad f_{x_i}(t) = p_i(t)^{a_i}, \quad m_{x_i}(t) = p_i(t)^{e_i}, \quad \text{and} \quad a_i \deg(p_i(t)) = n_i d.$$

This shows the following proposition.

**Proposition 5.** *Let  $f(t)$  be as in Theorem 4. Then  $f(t)$  is a characteristic polynomial of  $A$  if and only if for all  $i = 1, \dots, s$ ,  $a_i \deg(p_i(t)) = n_i d$  for some  $n_i \in \mathbb{N}$  and the polynomial  $p_i(t)^{a_i}$  is a characteristic polynomial of  $\text{Mat}_{n_i}(\Delta)$ .*

Therefore, it suffices to consider the case where  $f(t)$  is a power of an irreducible polynomial.

**Lemma 6.** *Let  $p(t) \in F[t]$  be a monic irreducible polynomial and put  $E := F[t]/(p(t))$ . Then a polynomial  $f(t)$  of the form  $p(t)^a$  of degree  $nd$  is a characteristic polynomial of  $A$  if and only if there is an  $F$ -algebra embedding of  $E$  in  $A$ .*

PROOF. Let  $\bar{t}$  be the image of  $t$  in  $E$  and suppose that there is an  $F$ -algebra embedding  $\rho : E \rightarrow A$ . Put  $x := \rho(\bar{t})$ . Then  $m_x(t) = p(t)$  and  $f_x(t) = p(t)^a$  by Lemma 2 (2). Conversely, suppose there is an element  $x \in A$  such that  $f(t) = f_x(t)$ . Let

$$0 = V_0 \subset V_1 \subset \dots \subset V_{l-1} \subset V_l = V$$

be a maximal chain of  $F[x]$ -invariant  $\Delta$ -subspaces. The minimal polynomial of  $x$  on each factor  $V_i/V_{i-1}$  is equal to  $p(t)$ . Choosing an  $\Delta$ -linear isomorphism  $\rho : \bigoplus_{i=1}^l V_i/V_{i-1} \simeq V$ , we get an element  $x^{ss} := \rho \circ x \circ \rho^{-1}$  (called a semi-simplification of  $x$ ) in  $\text{End}_\Delta(V)$  whose minimal polynomial is equal to  $p(t)$ . The map  $\bar{t} \mapsto x^{ss}$  gives an  $F$ -embedding of  $E$  in  $A$ . ■

When  $F$  is a global field and the degree  $[E : F] = \deg(A)$  is maximal, one can use the local-global principle to check the condition in Lemma 6. However, when the degree  $[E : F]$  is not maximal, the local-global principle for embedding  $E$  in  $A$  over  $F$  fails in general; see constructions of counterexamples in [7, Sections 4 and 5]. The following lemma provides an alternative method to check this condition.

**Lemma 7.** *Let  $E$  be as in Lemma 6. There is an  $F$ -algebra embedding of  $E$  in  $A$  if and only if  $[E : F] \mid n \cdot c(\Delta \otimes_F E)$ .*

PROOF. This is a special case of [8, Theorem 2.9]. We provide a direct proof for the reader's convenience. Write  $\Delta \otimes_F E = \text{Mat}_c(\Delta')$ , where  $\Delta'$  is the division part of the central simple algebra  $\Delta \otimes_F E$  over  $E$ . An  $F$ -algebra embedding of  $E$  into  $A = \text{Mat}_\Delta(V)$  exists if and only if  $V$  is an  $(E, \Delta)$ -bimodule, or equivalently a right  $E \otimes_F \Delta = \text{Mat}_c(\Delta')$ -module. By the dimension counting, the vector space  $V$  is a  $\text{Mat}_c(\Delta')$ -module if and only if

$$(4.2) \quad \frac{\dim_F V}{c[\Delta' : F]} \in \mathbb{N}.$$

Note that  $[E : F][\Delta : F] = c^2[\Delta' : F]$ . From this relation and that  $\dim_F V = n[\Delta : F]$ , the condition (4.2) can be written as  $[E : F] \mid nc$ . This proves the lemma. ■

By Proposition 5 and Lemmas 6 and 7, the proof of Theorem 4 is complete. ■

## 5. CONJUGACY CLASSES

Let  $x \in A$  be an separable element as in Section 3. Suppose we have another element  $x' \in A$  with the same characteristic polynomial  $f_{x'}(t) = f_x(t) = \prod_{i=1}^s p_i(t)^{a_i}$  and minimal polynomial  $m_{x'}(t) = m_x(t) = \prod_{i=1}^s p_i(t)^{e_i}$ . We may identify  $F[x'] =$

$F[t]/(m_x(t)) = F[x]$ . The  $(F[x'], \Delta)$ -bimodule structure on  $V$  gives a similar decomposition into  $\Delta$ -submodules  $V'_i$  as (3.1)

$$(5.1) \quad V = \bigoplus_{i=1}^s V'_i, \quad \dim_{\Delta} V'_i =: n'_i > 0, \quad \sum_{i=1}^s n'_i = n.$$

Similarly, we also have  $V'_i = (W'_i)^{\oplus c_i}$ , where  $W'_i$  is a  $\tilde{\Delta}_i$ -module, and there is an isomorphism of  $\tilde{\Delta}_i$ -modules

$$(5.2) \quad W'_i \simeq \bigoplus_{j=1}^{t'_i} \Delta_i[\varepsilon_i]/(\varepsilon_i^{m'_j}).$$

Note that  $n_i = n'_i$  as  $n_i d = a_i \deg p_i(t) = n'_i d$  (see (4.1)). The elements  $x$  and  $x'$  are conjugate by an element in  $A^\times$  if and only if the  $(F[x], \Delta)$ -bimodule structure and  $(F[x'], \Delta)$ -bimodule structure on  $V$  are equivalent. This is equivalent to that

$$(5.3) \quad t_i = t'_i, \quad \text{and} \quad (m_1, \dots, m_{t_i}) = (m'_1, \dots, m'_{t_i}), \quad \forall i = 1, \dots, s.$$

The tuple  $(m_1, \dots, m_{t_i})$  is a partition of the integer  $\dim_{\Delta_i}(W_i)$ . This integer is given by the following proposition.

**Proposition 8.** *We have  $\dim_{\Delta_i} W_i = n_i c_i / \deg p_i(t) = a_i / \deg(\Delta_i)$ .*

PROOF. Write  $\Delta \otimes s_i(F_i) = \text{Mat}_{c_i}(\Delta_i) = \text{End}_{\Delta_i}(L_i)$  for a right  $\Delta_i$ -vector space  $L_i$ . As  $\Delta \subset \Delta \otimes_F s_i(F_i)$  and  $L_i$  is a  $\Delta \otimes s_i(F_i)$ -module, so  $L_i$  is a  $\Delta$ -module. It follows that  $(\dim_F L_i) / [\Delta : F] \in \mathbb{N}$ . Put  $d_i = \deg(\Delta_i)$ . We have  $d = c_i d_i$  and (using  $\dim_F L_i = c_i [\Delta_i : F]$ )

$$(5.4) \quad c_i [\Delta_i : F] / [\Delta : F] = c_i d_i^2 \deg p_i / d^2 = \deg p_i / c_i \in \mathbb{N}.$$

Now we have

$$(5.5) \quad \dim_{\Delta_i} W_i = \frac{\dim_F V_i}{c_i [\Delta_i : F]} = \frac{[\Delta : F] \dim_{\Delta} V_i}{c_i [\Delta_i : F]} = \frac{n_i c_i}{\deg p_i} = \frac{d n_i c_i}{d \deg p_i} = \frac{a_i}{d_i}.$$

This proves the proposition.  $\blacksquare$

Let  $S$  be the set of monic separable irreducible polynomials  $p$  of  $F[t]$  with  $p \neq t$ . For each irreducible polynomial  $p$  in  $F[t]$ , let  $\Delta_p$  be the division part of the central simple algebra  $\Delta \otimes_F k(p)$  over  $k(p)$ , where  $k(p)$  is the residue field at the prime  $(p)$ . A partition  $\lambda = (\lambda_1, \dots, \lambda_t)$  is a non-decreasing positive integers, and we write  $|\lambda| := \sum_{i=1}^t \lambda_i$ . Put  $|\lambda| = 0$  if  $\lambda = \emptyset$ .

**Theorem 9.** *The association from  $x$  to its characteristic polynomial  $f_x(t)$  and the partitions of the integers  $\dim_{\Delta_i} W_i$  by (3.2) induces a bijection between the set of separable conjugacy classes of the multiplicative group  $A^\times$  and the set of partition-valued functions  $\lambda$  on  $S$  such that*

$$(5.6) \quad \sum_{p \in S} \deg(p) |\lambda(p)| \deg(\Delta_p) = \deg(A).$$

PROOF. We have shown the injectivity (see (5.3)). We show the surjectivity. Let  $p_1, \dots, p_s$  be those with  $|\lambda(p_i)| \neq 0$ . Let  $d_i := \deg(\Delta_{p_i})$  and  $a_i := |\lambda(p_i)| d_i$ . We need to show that the conditions (a) and (b) of Theorem 4 for the polynomial  $\prod_{i=1}^s p_i^{a_i}$  are satisfied. But these conditions are satisfied due to

$$n_i c_i / \deg p_i = a_i / d_i = |\lambda(p_i)| \in \mathbb{N}$$

and (see (5.4))

$$a_i \deg p_i / d = (a_i / d_i)(\deg p_i / c_i) \in \mathbb{N}.$$

This proves the theorem. ■

When  $A = \Delta$  is a division algebra, any minimal polynomial of  $\Delta$  is irreducible. The same proof presented here shows that the set of all conjugacy classes of  $A^\times$  is parametrized by monic irreducible polynomials  $p \neq t$  such that  $\deg p \cdot \deg(\Delta_p) = \deg(\Delta)$ .

When  $\text{char } F = 0$ , for example if  $F$  is a number field, Theorem 9 gives a complete description of conjugacy classes of the multiplicative group  $A^\times$ .

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